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Riemann Rearrangement Theorem for some types of convergence

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ABSTRACT

We reexamine the Riemann Rearrangement Theorem for different types of convergence and classify possible sum ranges for statistically convergent series and for series that converge along the $2n$-filter.

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1. Introduction

The classical Riemann Rearrangement Theorem (RRT for short) says that the commutative law is no longer true for infinite sums. To be more precise it says the following:

Theorem 1.0.1 *(RRT). Let $\sum_{k=1}^{\infty} x_k$ be a conditionally convergent series of real numbers. Then*:

(1) *for any $s \in \mathbb{R}$ one can find a permutation π such that $\sum_{k=1}^{\infty} x_{\pi(k)} = s$;*
(2) *one can find a permutation σ such that $\sum_{k=1}^{\infty} x_{\sigma(k)} = \infty$;*
(3) *one can find a permutation σ such that $\sum_{k=1}^{\infty} x_{\sigma(k)} = -\infty$.*

In the RRT one considers the ordinary convergence of series. It looks natural to consider in this setting some weaker types of convergence. Interesting results in this direction are proved in [1] and [6], where generalizations of the Riemann theorem for Cesaro summation and other matrix summation methods were obtained. These generalizations are much more complicated than the original Riemann theorem, and even the statements strongly differ from the classical one: for Cesaro summation it is possible for the set of sums under all permutations of summands to form an arithmetic progression. V. Kadets posed the problem of what effects appear if the ordinary convergence in the statement of the Riemann theorem is substituted by convergence with respect to a filter. In this paper we do two steps in this direction, namely we consider statistical convergence and convergence of the subsequence $\sum_{k=1}^{2n} x_k$ of partial sums.

In this paper natural numbers $\mathbb{N}$ start from 1.

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2. Statistical convergence

2.1. Introduction

Statistical convergence is a generalization of the usual notion of convergence that parallels the classical theory. While statistical convergence has become an active area of research under the name of statistical convergence only recently, it appeared in the literature some time ago. This notion goes back at least to the work of H. Fast (see [3]).

Statistical convergence is used in the number theory, trigonometric series and summability theory. A relation between statistical convergence and Banach space theory, as well as the list of references, can be found in [2]. The aim of this section is to generalize RRT to the case of the statistical convergence.

The object that is going to be investigated is $SR_{\mathrm{st.}}(\sum x_k)$ and the sequence of definitions below leads to it.

Definition 2.1.1. A set $A \subset \mathbb{N}$ is said to be of *density zero* if

$$\lim_{n\to\infty} \frac{|A \cap \{1,\dots,n\}|}{n} = 0.$$

A set $A \subset \mathbb{N}$ is said to be of density one if its complement $\mathbb{N} \setminus A$ is of density zero.

Definition 2.1.2. A sequence $\{s_n\}_{n=1}^{\infty}$ *statistically converges* to s (notation: $s_n \xrightarrow{\text{stat.}} s$) if for every $\varepsilon > 0$ the set $\{n\colon |s_n - s| > \varepsilon\}$ is of density zero.

Definition 2.1.3. Series $\sum x_k$ is said to be *convergent statistically* to s if the sequence $s_n = \sum_1^n x_k$ of partial sums converges statistically to s (short notation is $\sum x_k \overset{\text{st.}}{=} s$).

Definition 2.1.4. A point s belongs to the *statistical sum range* of the series $\sum x_k$ if there exists a permutation π such that $\sum_{k=1}^n x_{\pi(k)} \xrightarrow{\text{stat.}} s$. The set of all such points is called the *statistical sum range* of the series and is denoted by $SR_{\mathrm{st.}}(\sum x_k)$.

We will use also the following definition from [5].

Definition 2.1.5. A point x is said to be a *limit point* for the series $\sum x_k$ if it is the limit point of some subsequence of the sequence of partial sums of some rearrangement of the series, i.e.,

$$\exists \pi\ \exists \{n_k\}_{k=1}^{\infty}\colon \quad \sum_{i=1}^{n_k} x_{\pi(i)} \to x.$$

The set of all such points is called the *limit-point range* of the series and is denoted by $\mathrm{LPR}(\sum x_k)$.

It is known that $\mathrm{LPR}(\sum x_k)$ is a closed set (see [4, p. 73]) and $SR_{\mathrm{st.}}(\sum x_k) \subset \mathrm{LPR}(\sum x_k)$. H. Hadwiger [4] proved that $\mathrm{LPR}(\sum x_k)$ is a shifted closed additive subgroup of the space in which the series lives. In particular this is true for numerical series (see also [5], Exercises 3.2.2, 2.1.2 and comments to these exercises).

By $\overline{\mathbb{R}}$ we denote the two-point compactification of the real line:

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}.$$

2.2. Main theorem for $SR_{\mathrm{st.}}$

The aim of this section is to prove the following result:

Theorem 2.2.1. *Let $\sum x_k \overset{\text{st.}}{=} a$ for the original ordering of x_i. Then $SR_{\mathrm{st.}}(\sum x_k) = \mathrm{LPR}(\sum x_k)$. So $SR_{\mathrm{st.}}(\sum x_k)$ is one of the following*:

(1) *The only number a*;
(2) $\{a + \lambda\mathbb{Z}\}$ *for some* $\lambda \in \mathbb{R}$;
(3) *The whole of* $\mathbb{R}$.

Proof. Since the series $\sum x_k$ converges statistically there exists a subsequence x_{n_k} such that $x_{n_k} \to 0$. From the elements of x_{n_k} we can select a subsequence $x_{n_{k_i}}$ such that $\sum_{i=1}^{\infty} |x_{n_{k_i}}| < \infty$.

Now we can substitute 0 for all the elements $x_{n_{k_i}}$ in the original series and this will not affect the convergence since we are subtracting an absolutely convergent series. So without loss of generality we may assume that there are infinitely many zeros among the original series terms.

Let us write the definition of LPR in detail:

$$\mathrm{LPR}\Bigl(\sum x_k\Bigr) = \left\{x\colon\ \exists\pi\ \exists\{m_k\}\ x = \lim_{k\to\infty}\sum_{j=1}^{m_k} x_{\pi(j)}\right\}$$

where π is a permutation of $\mathbb{N}$ and $\{m_k\}$ is an increasing sequence of indices. Let b be an arbitrary element of LPR. Let $\{m_k\}$ be a sequence from the definition corresponding to the element b. We arrange elements of our series in the following way:

$$\underbrace{0+\cdots+0}_{(m_1)^2\text{ times}}+x_{\pi(1)}+\cdots+x_{\pi(m_1)}+\underbrace{0+\cdots+0}_{(m_2)^2\text{ times}}+x_{\pi(m_1+1)}+\cdots+x_{\pi(m_2)}+\underbrace{0+\cdots+0}_{(m_3)^2\text{ times}}+\cdots.$$

Note that this gives a permutation (which is different from π) of the original sequence: by the assumption there were infinitely many zeroes among x_i's, and we added countably many more, so still have infinitely many.

We claim that this permuted series statistically converges to b. Indeed, let B denote the set of indices of added zeroes. Then the lower density of B (we thank the referee for this calculation) equals to

$$\lim_{k\to\infty}\frac{\sum_{j=1}^k m_j^2}{\sum_{j=1}^k m_j+\sum_{j=1}^k m_j^2}\overset{\text{Stolz}}{=}\lim_{k\to\infty}\frac{m_k^2}{m_k+m_k^2}=1,$$

and so B is of density 1. Since the above series converges to b along B, the claim follows. □

2.3. Examples

We finish the proof by giving examples which satisfy cases of Theorem 2.2.1.

Example 2.3.1. Any unconditionally convergent series in the classical meaning gives us a series with $SR_{\mathrm{st.}}=\{a\}$, which corresponds to the case (1).

Example 2.3.2. Take $\lambda\in\mathbb{R}$. Let the elements of the series be the following:

$$x_n=\begin{cases}0, & n\neq 10^k \text{ and } n\neq 10^k+1,\\ \lambda, & n=10^k,\\ -\lambda, & n=10^k+1;\end{cases}\qquad k\in\mathbb{N},\ n\in\mathbb{N}.$$

Then $SR_{\mathrm{st.}}=\lambda\mathbb{Z}$, which corresponds to the case (2).

Example 2.3.3. Any conditionally convergent series in the classical meaning gives us a series with $SR_{\mathrm{st.}}=\mathbb{R}$, which corresponds to the case (3).

Remark 2.3.4. In fact the statement $SR_{\mathrm{st.}}(\sum x_k)=\mathrm{LPR}(\sum x_k)$ holds for any series in any Banach space. Thus, one can prove that in any separable Banach space, $SR_{\mathrm{st.}}$ can be any shifted closed subgroup.

Remark 2.3.5. If one wants to consider $SR_{\mathrm{st.}}\subset\overline{\mathbb{R}}$, then modifying the argument above one can show:

Theorem 2.3.6. *Let $\sum x_k\overset{\mathrm{st.}}{=}a$ for the original permutation. Then $SR_{\mathrm{st.}}(\sum x_k)$ is one of the following*:

(1) *The only number a*;
(2) $\{a+\lambda\mathbb{Z}\}\cup\{-\infty,\infty\}$ *for some* $\lambda\in\mathbb{R}$;
(3) *The whole of* $\overline{\mathbb{R}}$;
(4) *The set* $\{-\infty,a,\infty\}$.

An example of a series of the fourth type would be the following series $\sum x_i$ (we thank the referee for this particular example). Define inductively $a_1:=1$ and $a_n:=n+\sum_{k=1}^{n-1}a_k$ for $n\geqslant 2$; then let $x_{2i-1}:=(-1)^{i+1}a_i$ and $x_{2i}:=-x_{2i-1}$.

3. 2*n*-convergence

3.1. Introduction

Let's say that a series $\sum_{k=1}^{\infty}x_k$ $2n$-converges to c if

$$\lim_{n\to\infty}\sum_{k=1}^{2n}x_k=c.$$

Definition 3.1.1. A point $s \in \mathbb{R}$ belongs to the $2n$ *sum range* of the series $\sum x_k$ if there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$\lim_{n\to\infty} \sum_{k=1}^{2n} x_{\pi(k)} = s.$$

The set of all such points is called the $2n$ *sum range* of the series $\sum x_k$ and is denoted by $SR_2(\sum x_k)$. When it is clear which series is considered, we will denote this set just by SR_2.

Let us first consider the following example:

Example 3.1.2. The series

$$1 + (-1) + 1 + (-1) + \cdots$$

obviously diverges in the classical sense. But if one takes the subsequence $S_n = \sum_{k=1}^{2n} x_k$ of its partial sums, then $S_n = 0$ for all n and so this subsequence converges. Now permute terms of this series. Note that in order to converge, after a certain (necessarily even) step elements must go in strict pairs $1 + (-1)$. For example, one has

(1) $1 + 1 + 1 + (-1) + 1 + (-1) + \cdots = 2$, or
(2) $(-1) + (-1) + 1 + (-1) + 1 + (-1) + \cdots = -2$,

and it is easy to prove that

$$\left\{ S \in \mathbb{R}\colon \exists \pi \ \lim_{n\to\infty} \sum_{k=1}^{2n} x_{\pi(k)} = S \right\} = 2\mathbb{Z}.$$

So the statement of the Riemann Rearrangement Theorem for the case of $2n$-convergence has to be modified. Surprisingly, this modification and its proof turn out to be rather non-trivial and more complicated than in the case of statistical convergence.

Recall that the set $X \subset \mathbb{R}$ is said to be ε*-separated* if all pairwise distances between the elements of X are greater than ε. X is said to be *separated* if it is ε-separated for some $\varepsilon > 0$.

The aim of this chapter is to prove the following result:

Theorem 3.1.3 *(Main theorem). Let* $\lim_{n\to\infty} \sum_{k=1}^{2n} x_k = a \in \mathbb{R}$. *Then* $SR_2(\sum x_k)$ *is one of the following*:

(1) *A shifted additive subgroup of the form*

$$a + \left\{ c_1 z_1 + \cdots + c_l z_l\colon \ z_k \in E, \ c_i \in \mathbb{Z}, \ \sum_{k=1}^{l} c_k \text{ is even}, \ l \in \mathbb{N} \text{ is not fixed} \right\},$$

where E *is a separated set*;
(2) *The whole of* $\mathbb{R}$;
(3) *The real number* a.

3.2. Reduction to a special form of the series

Any series $\sum x_k$ can be written as

$$x_1 + (-x_1 + \alpha_1) + x_3 + (-x_3 + \alpha_2) + x_5 + (-x_5 + \alpha_3) + \cdots \tag{1}$$

by setting

$$\alpha_k \stackrel{\text{def}}{=} x_{2k-1} + x_{2k} \quad (\forall k \in \mathbb{N}).$$

Hence if the series $\sum x_k$ $2n$-converges in the original order, i.e. $\lim_{n\to\infty} \sum_{k=1}^{2n} x_k = a$, then $\sum_{i=1}^{\infty} \alpha_i = a$.

Theorem 3.2.1. *If* $\sum_{k=1}^{\infty} \alpha_k$ *converges conditionally, then*

$$SR_2\left(\sum x_k\right) = \mathbb{R}. \tag{2}$$

Proof. The RRT says that for a conditionally convergent series $\sum_{k=1}^{\infty}\alpha_k$, for all $c\in\mathbb{R}$ there exists a permutation of indices π such that $\sum_{k=1}^{\infty}\alpha_{\pi(k)}=c$. Consider the following arrangement of $\{x_k\}$:

$$x_{2\pi(1)-1}+(-x_{2\pi(1)-1}+\alpha_{\pi(1)})+x_{2\pi(2)-1}+(-x_{2\pi(2)-1}+\alpha_{\pi(2)})+\cdots.$$

It's clear that this series $2n$-converges to c. As c was arbitrary, we get (2). $\square$

Definition 3.2.2. A series $\sum_k x_k$ is said to be *equivalent* to $\sum_k y_k$ if $\sum_k|x_k-y_k|<\infty$.

Remark that if one of two equivalent series converges ($2n$-converges) in some permutation then so does the second series. Note also that $SR_2(\sum_k x_k)=SR_2(\sum_k y_k)+\sum_k(x_k-y_k)$.

Theorem 3.2.1 corresponds to the case (2) of the main theorem. Now consider what happens if $\sum_{k=1}^{\infty}\alpha_k$ converges unconditionally to a. In this case $\sum_k x_k$ is equivalent to the following simplified series:

$$x_1+(-x_1)+x_3+(-x_3)+x_5+(-x_5)+\cdots. \tag{3}$$

So we reduce the series (1) to (3). Now since $SR_2(\sum x_n)=SR_2(\sum x_{\pi(n)})$ for every series and every permutation π, we may consider a series of the form:

$$x_1+x_{-1}+x_2+x_{-2}+x_3+x_{-3}+\cdots \tag{4}$$

where $x_{-n}=-x_n$ and $x_n\geqslant 0$ for $n>0$, $x_n\leqslant 0$ for $n<0$. Denote by X the set of all values of elements of the series

$$X=\{c\colon \exists n\in\mathbb{Z}\setminus\{0\}\ x_n=c\}.$$

By the order of an element $e\in X$ we mean

$$\chi(e)=\bigl|\{i\in\mathbb{Z}\setminus\{0\}\colon x_i=e\}\bigr|.$$

3.3. The (basic) case of separated X

Lemma 3.3.1. *Let X be ε-separated and suppose it contains nonzero elements of infinite order. Then*

$$SR_2=\left\{c_1e_{j_1}+\cdots+c_re_{j_r}\colon e_{j_k}\in X,\ \chi(e_{j_k})=\infty,\ c_k\in\mathbb{Z},\ \sum_{j=1}^{r}c_j \text{ is even},\ r \text{ is not fixed}\right\}. \tag{5}$$

If there are no nonzero elements of infinite order, then

$$SR_2=\{0\}.$$

Proof. Denote the right-hand side of (5) by $\mathfrak{L}$. Let us prove that $SR_2\subset\mathfrak{L}$.

Let $\pi:\mathbb{N}\to\mathbb{Z}\setminus\{0\}$ be an arbitrary bijection such that

$$A=\lim_{n\to\infty}\sum_{j=1}^{2n}x_{\pi(j)}\in\mathbb{R}.$$

By the Cauchy criterion there exists an even number n_0 such that for every even n and m greater or equal than n_0 the following inequality holds:

$$|S_n-S_m|<\varepsilon.$$

Take and even number $n>n_0$. Then

$$|S_{n+2}-S_n|=|x_{\pi(n+1)}+x_{\pi(n+2)}|<\varepsilon.$$

But X is ε-separated and hence $|x_{\pi(n+1)}+x_{\pi(n+2)}|=0$. Thus,

$$\bigl(|x_{\pi(n+1)}+x_{\pi(n+2)}|=0\bigr)\quad\Leftrightarrow\quad(x_{\pi(n+1)}=-x_{\pi(n+2)}).$$

It follows that the series has the following structure

$$A=x_{\pi(1)}+\cdots+x_{\pi(n_0)}+x_{k_1}+(-x_{k_1})+x_{k_2}+(-x_{k_2})+\cdots.$$

So eventually terms come in strict pairs and thus $S_n=S_{n_0}$ for any even $n>n_0$.

Now in the sum $x_{\pi(1)}+\cdots+x_{\pi(n_0)}$ consider the elements $y_1,y_2,\ldots,y_j$ of *finite* order. Then we claim for any $i\in\{1,\ldots,j\}$:

$$\bigl|\{k\colon 1\leqslant k\leqslant n_0,\ x_{\pi(k)}=y_i\}\bigr|=\bigl|\{k\colon 1\leqslant k\leqslant n_0,\ x_{\pi(k)}=-y_i\}\bigr|.$$

This is so since in the second part of the series all elements come in strict pairs. And so elements of finite order cancel each other. The only elements left are thus the elements of *infinite* order. So A has the requested form

$$A = c_1 e_{j_1} + \cdots + c_r e_{j_r},$$

where $c_i \in \mathbb{Z}$. Moreover, since n_0 was even, $\sum_{k=1}^{l} c_k$ is even.

We now show that $\mathfrak{L} \subset SR_2$.

Select an element $z \in \mathfrak{L}$, $z = c_1 e_{j_1} + \cdots + c_r e_{j_r}$ and define the series starting with

$$\operatorname{sign}(c_1)(\underbrace{e_{j_1} + \cdots + e_{j_1}}_{|c_1| \text{ times}}) + \cdots + \operatorname{sign}(c_r)(\underbrace{e_{j_r} + \cdots + e_{j_r}}_{|c_r| \text{ times}})$$

and after these terms the rest of the series is

$$x_{k_1} + (-x_{k_1}) + x_{k_2} + (-x_{k_2}) + \cdots.$$

It is obvious that the $2n$-sum of this series is z and that the series is a rearrangement of (3). This shows that $\mathfrak{L} \subset SR_2$. $\square$

3.4. Some combinatorial lemmas

Let M be a set of indices. For a bounded sequence $(x_n)_{n\in M}$, we define $\Delta(M)$ by

$$\Delta(M) = \Delta\big(M, (x_n)_{n\in M}\big) = \inf_{a\in\mathbb{R}} \sum_{n\in M} |x_n - a|. \tag{6}$$

We also define the sum of an empty set of summands to be 0.

Lemma 3.4.1. *If $\Delta(M) = \infty$, then one can find a finite collection of disjoint pairs $n_k, m_k \in M$, $k = 1, 2, \ldots, s$ such that $\sum_{k=1}^{s} |x_{n_k} - x_{m_k}|$ is arbitrarily large.*

Proof. Suppose that $\{x_n\}_{n\in M}$ has only one limiting point (otherwise the statement is obvious). Denote it by a. The fact that $\Delta(M) = \infty$ implies that

$$\sum_{n\in M} |x_n - a| = \infty.$$

For every $K > 0$ and every $\delta > 0$ there is s such that

$$\sum_{k=1}^{s} |x_{n_k} - a| > K + \delta,$$

where $n_k = k$. We can then select a subsequence $\{x_{m_k}\}$ disjoint from $\{x_{n_k}\}$ such that

$$\sum_{k=1}^{s} |x_{m_k} - a| < \delta.$$

This can be done since a is the limit point of the sequence. The subsequence obtained now satisfies $\sum_{k=1}^{s} |x_{n_k} - x_{m_k}| > K$. Since K was arbitrary, this proves the lemma. $\square$

Lemma 3.4.2. *If $\Delta(M) < \infty$, then*:

(1) *Either M is finite, or $(x_n)_{n\in M}$ has only one limiting point.*
(2) *The quantity $\sum_{n\in M} |x_n - a|$ in (6) attains its minimum at the point $a(M)$. If M is infinite, then there is the only possibility for $a(M)$: it must be the only limiting point of $(x_n)_{n\in M}$. If M is finite then $a(M)$ can be any median of $(x_n)_{n\in M}$, i.e. a point a with the following property*:

$$\big|\{n \in M\colon x_n < a\}\big| = \big|\{n \in M\colon x_n > a\}\big|.$$

(3) *For every $\varepsilon > 0$ one can find a finite collection of disjoint pairs $n_k, m_k \in M$, $k = 1, 2, \ldots, s$ such that*

$$\sum_{k=1}^{s} |x_{n_k} - x_{m_k}| > \Delta(M) - \varepsilon. \tag{7}$$

Proof. (1): The statement is obvious.

(2): Assume first that M is infinite. Since $\Delta(M) < \infty$ there is at least one point a such that $\sum_{n\in M}|x_n - a| < \infty$. Then $x_n - a$ tends to 0 along M, so a is the only limiting point of $(x_n)_{n\in M}$. The case of finite M is obvious.

(3): We first deal with a finite M. In this case we can chose x_{n_1} to be the leftmost element with respect to $a(M)$ and x_{m_1} to be the rightmost. We then define x_{n_2} as the leftmost element of the remaining terms x_{m_2} to be the rightmost, etc. We obtain that

$$\sum_{k=1}^{s}|x_{n_k} - x_{m_k}| = \Delta(M).$$

If M is infinite we proceed like in Lemma 3.4.1. $\square$

Let $G = \{G_k\}$, $k \in \mathbb{N}$, be a pairwise disjoint collection of subsets of $\mathbb{R}$. We say that G is an ε-collection, if the diameters of all the G_k do not exceed ε. Denote $M_k = \{n \in \mathbb{N}\colon\ x_n \in G_k\}$; $\Delta_G = \sum_{k\in\mathbb{N}} \Delta(M_k)$. Now the proof of the theorem splits into two cases.

3.5. Case 1 (reduction to the case of separated X)

By the distance between two sets A, B of real numbers we mean $d(A, B) = \inf\{|a - b|\colon\ a \in A,\ b \in B\}$. If one of A, B is empty, we set $d(A, B) = +\infty$.

Lemma 3.5.1. *Let $\{x_n\}$ have the following property: there is an $\varepsilon > 0$ such that $\Delta_G < \infty$ for every ε-collection G. Then the series $\sum x_n$ is equivalent to a series $\sum y_n$ with a separated set of elements (as in Lemma 3.3.1), so it satisfies the statement of the main theorem.*

Proof. Let ε satisfy the condition of the lemma. We are going to cover the set of values $X^+ = X \cap \mathbb{R}^+$ by an ε-collection G of intervals in such a way that there is an n_0 such that for all $n, m > n_0$, all the distances between G_n and G_m are bigger than $\frac{\varepsilon}{4}$. If such a G is selected, put $M_k = \{n \in \mathbb{N}\colon\ x_n \in G_k\}$ and denote $a_k = a(M_k) \in G_k$, $k \in \mathbb{N}$, the number from Lemma 3.4.2. In this case the sequence a_k is separated, and we can define the required symmetric sequence $y_n, n \in \mathbb{Z}\setminus\{0\}$, as follows: $y_n = \operatorname{sign}(n)a_k$ for $|n| \in M_k$. The set of elements of $\sum y_n$ equals $\{a_1, -a_1, a_2, -a_2, \ldots\}$, so it is separated, and the mutual equivalence of $\sum x_n$ and $\sum y_n$ follows from the inequality $\sum_k |x_k - y_k| \leqslant 2\Delta_G < \infty$. So all what we need is to construct a G with the property described above.

Consider covering of X^+ by $T_k = X^+ \cap [(k-1)\varepsilon, k\varepsilon)$ and set $t_k = \inf T_k$, $t^k = \sup T_k$ if $T_k \neq \emptyset$ and $t_k = t^k = (k - 1/2)\varepsilon$ if $T_k = \emptyset$. Since $T = (T_k)_{k\in\mathbb{N}}$ forms an ε-collection, we have

$$\sum_k \left|t^k - t_k\right| \leqslant 2\Delta_T < \infty,$$

so in particular $|t^k - t_k| \to 0$. Select the required n_0 in such a way that $|t^k - t_k| < \frac{\varepsilon}{4}$ for all $k > n_0$. For $k \leqslant n_0$ put $G_k = [(k-1)\varepsilon, k\varepsilon)$. Before defining G_k for $k > n_0$ let us explain the picture. We would like to take $G_k = [t_k, t^k]$, but this can be a wrong selection, because for some k both t^k and t_{k+1} can be very close to $k\varepsilon$ and $t_{k+1} - t^k$ can be smaller than $\frac{\varepsilon}{4}$. But for such “bad” values of k the segment $[t_k, t^{k+1}]$ is of the length at most $\frac{3\varepsilon}{4}$, covers both the segments $[t_k, t^k]$ and $[t_{k+1}, t^{k+1}]$, and has at least distance $\frac{\varepsilon}{4}$ from the rest of $[t_j, t^j]$. So the required selection of G_k for $k > n_0$ can be done as follows: take all those segments $[t_j, t^j]$, $j > n_0$, which are far from the others (i.e., the distances to the others are bigger than $\frac{\varepsilon}{4}$), and add all those segments $[t_j, t^{j+1}]$, $j > n_0$, where $t_{j+1} - t^j < \frac{\varepsilon}{4}$. $\square$

3.6. The remaining case

Lemma 3.6.1. *Suppose now that for a sequence $\{x_n\}$ for every $\varepsilon > 0$ there is an ε-collection G such that $\Delta_G = \infty$. Then one can find a collection of disjoint pairs $n_k, m_k \in \mathbb{N}$, $k = 1, 2, \ldots$ such that $|x_{n_k} - x_{m_k}| \to 0$ as $k \to \infty$ and*

$$\sum_{k=1}^{\infty}|x_{n_k} - x_{m_k}| = \infty. \tag{8}$$

In this case $SR_2(\sum x_k) = \mathbb{R}$. This proves the statement of the main theorem.

Proof. For $\varepsilon = 1$ we can find an ε-collection G such that $\Delta_G = \infty$. Then applying Lemma 3.4.1 or (3) of Lemma 3.4.2 one can find a collection of disjoint pairs $n_k, m_k \in \mathbb{N}$, $k = 1, 2, \ldots, N_1$ such that $|x_{n_k} - x_{m_k}| < 1$ and $\sum_{k=1}^{N_1}|x_{n_k} - x_{m_k}| > 1$. For $\varepsilon = 1/2$ we select disjoint pairs $n_k, m_k \in \mathbb{N}$, $k = N_1 + 1, \ldots, N_2$ such that $|x_{n_k} - x_{m_k}| < 1/2$ and $\sum_{k=1}^{N_2}|x_{n_k} - x_{m_k}| > 2$. We proceed for $\varepsilon = 1/4$ and so on. To see that $SR_2(\sum x_k) = \mathbb{R}$, we consider the pairs

$$(x_{n_1} - x_{m_1}),\ (x_{m_1} - x_{n_1}),\ (x_{n_2} - x_{m_2}),\ (x_{m_2} - x_{n_2}),\ \ldots.$$

We add missing pairs of the form $x_i - x_i$ to include all the elements into the series. Permuting pairs (like in the RRT), we obtain $SR_2(\sum x_k) = \mathbb{R}$. $\square$

3.7. Examples

To complete the paper we are going to demonstrate that for each of the cases (1)–(3) of the main Theorem 3.1.3 there exists a series satisfying it. To write down such examples, let us introduce a more compact way of writing the series

$$\mathfrak{S} \stackrel{\text{def}}{=} \{(y_i, n_i)\colon\ y_i \in \mathbb{R},\ n_i \in \mathbb{N} \cup \{\infty\}\},$$

where n_i corresponds to the number of copies of y_i we have in the simplified series (n_i is the order of the element y_i) and the following condition is satisfied

$$y_i \neq y_j \quad (\forall i \neq j).$$

Say,

$$SR_2\big((1, \infty), (-1, \infty)\big) = SR_2\big(1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots\big).$$

General example: for any ε-separated set $E = \{e_i\}_{i=1}^{\infty}$ and $\mathfrak{S} = \{(e_i, \infty)\}$ we have

$$SR_2(\mathfrak{S}) = \left\{c_1 e_1 + \cdots + c_l e_l\colon\ e_k \in E,\ c_i \in \mathbb{Z},\ \sum_{k=1}^{l} c_k \text{ is even}\right\}.$$

In particular consider the following two examples.

Example 3.7.1. $\mathfrak{S}_1 = \{(1, \infty), (-1, \infty)\}$, e.g., the series $1 - 1 + 1 - 1 + \cdots$.
Here we get that $SR_2(\mathfrak{S}_1) = \{2\mathbb{Z}\}$. Notice that $2\mathbb{Z}$ is 2-*separated*.

Example 3.7.2. $\mathfrak{S}_2 = \{(1, \infty), (-1, \infty), (\sqrt{2}, \infty), (-\sqrt{2}, \infty)\}$, e.g.,

$$1 - 1 + \sqrt{2} - \sqrt{2} + 1 - 1 + \sqrt{2} - \sqrt{2} + \cdots.$$

Applying Lemma 3.3.1 here we get that

$$SR_2(\mathfrak{S}_2) = \{a \cdot 1 + b \cdot \sqrt{2}\},$$

where $(a + b)$ is even. It's obvious that $SR_2(\mathfrak{S}_2)$ is dense in $\mathbb{R}$.

Example 3.7.3. Let $\mathfrak{S}_3$ be any conditionally convergent series (in the usual sense). Then $SR_2(\mathfrak{S}_3) = \mathbb{R}$.

Example 3.7.4. $\mathfrak{S}_4 = \{(S_n, 1), (-S_n, 1)\colon\ n \in \mathbb{N}\}$, where $S_n = \sum_{i=1}^{n} \frac{1}{i}$, e.g., the series $S_1 - S_1 + S_2 - S_2 + \cdots$. For this series also $SR_2(\mathfrak{S}_4) = \mathbb{R}$ holds.

Example 3.7.5. Let $\mathfrak{S}_5$ be an any series unconditionally convergent to $a \in \mathbb{R}$ (in the usual sense). This series gives us $SR_2(\mathfrak{S}_5) = \{a\}$.

Remark 3.7.6. If one considers convergence in $\overline{\mathbb{R}}$, the main theorem has to be modified as follows:

Theorem 3.7.7. *Let* $\lim_{n\to\infty} \sum_{k=1}^{2n} x_k = a \in \mathbb{R}$. *Then* $SR_2(\sum x_k)$ *is one of the following*:

(1) *A shifted additive subgroup of the form*

$$a + \left\{c_1 z_1 + \cdots + c_l z_l\colon\ z_k \in E,\ c_i \in \mathbb{Z},\ \sum_{k=1}^{l} c_k \textit{ is even},\ l \in \mathbb{N}\right\} \cup \{-\infty, \infty\},$$

where E of is a separated set;

(2) *The whole of* $\overline{\mathbb{R}}$;

(3) *The real number* a;

(4) *The set* $\{-\infty, a, \infty\}$.

An example of the last case would be the series

$$1 - 1 + 2 - 2 + 4 - 4 + \cdots + 2^n - 2^n + \cdots.$$

3.8. Remark on 3n-convergence

At last we would like to make a remark on $3n$-convergence. We do not know what SR_3 can be, but it is easy to see that situation differs from the $2n$ case. Consider the following series (in the above notation) $\mathfrak{S} = \{(1/3, \infty), (10^i, 1), (-10^i - 1/3, 1){:}\ i \in \mathbb{N}\}$. It is the easy to see that $SR_3(\mathfrak{S}) = \mathbb{N} \cup \{0\}$. And so it is possible to have a sum range which is not a shifted subgroup of reals.

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References

[1] F. Bagemihl, P. Erdös, Rearrangements of C_1-summable series, Acta Math. 92 (1954) 35–53.
[2] J. Connor, M. Ganichev, V. Kadets, A characterization of Banach spaces with separable duals via weak statistical convergence, J. Math. Anal. Appl. 244 (2000) 251–261.
[3] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241–244.
[4] H. Hadwiger, Über das Umordnungsproblem im Hilbertschen Raum, Math. Z. 46 (1940) 70–79.
[5] M. Kadets, V. Kadets, Series in Banach Spaces: Conditional and Unconditional Convergence, Oper. Theory Adv. Appl., vol. 94, Birkhäuser, Basel, 1997, p. 36.
[6] G.G. Lorentz, K. Zeller, Series rearrangements and analytic sets, Acta Math. 100 (1958) 149–169.